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Channel flow induced by a travelling thermal wave

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Experiments conducted elsewhere show that a mean fluid motion can be induced in a channel by a travelling thermal wave. An analysis is carried out, linearized under the assumption that the induced motion is slower than the speed of the heat source. The expression for the mean motion is obtained for any Prandtl number and circular frequency of the thermal wave, to complete the results presented by Davey (1967) for low and high frequency ranges.

In the problem of the flow between two parallel plates, it is found that with a temperature profile symmetric about the centre of the channel, the induced flow does not exert a net shear force on either plate, while with a non-symmetric one, the plates are subjected to equal and opposite forces.

For the problem that the upper surface of the fluid is free and thermally insulated, an approximated result can be deduced from that of the previous problem by a simple transformation. It should agree with the result of Davey, obtained through a more elaborate procedure, except in the low frequency range when the surface deformation becomes important.

In agreement with the experiments, our analysis indicates that the induced mean motion is always in a direction opposite to that of the thermal wave, and its magnitude increases rapidly with decreasing Prandtl number. According to the theory, some of the previous experiments were not conducted under the optimum situations, and improved experimental conditions are suggested.

1. Introduction

It was demonstrated in the laboratory that a mean motion can be produced in a liquid confined within an axisymmetric container by heating its bottom with a rotating flame. Using water as the working fluid, it was found in a cylindrical vessel by Fultz *et al.* (1959), and in a cylindrical annulus by Stern (1959), that within the fluid a slow net angular motion was established in the sense opposite to the motion of the heat source. Schubert & Whitehead (1969) showed that the mean motion can be increased by decreasing the Prandtl number of the fluid medium. Instead of water, they used mercury filling in a cylindrical annulus and proved that the rate of rotation of the liquid was several times greater than that of the flame. They suspected that this phenomenon may provide an explanation for the high velocities of apparent cloud formations in the upper atmosphere of Venus, which have been observed in ultraviolet photographs by Smith (1967). Ignoring the effects of the curvature and the side walls of the cylindrical annulus, Stern considered a two-dimensional model with the fluid contained between two horizontal plates subjected to a sinusoidal temperature wave travelling along the channel. A linearized analysis was carried out corresponding to a special case where the thermal conductivity of the fluid is infinite, or the Prandtl number is zero. An improved model with finite Prandtl number was examined by Davey (1967). His work also includes a model with the fluid bounded by a plate below and a free surface above, used to simulate the experimental flow conditions.

When the induced fluid motion is faster than the travelling speed of the heat source, as in the experiment of Schubert & Whitehead, the linearized analysis cannot be applied. Under the assumption of vanishing Prandtl number for a fluid between two plates, Schubert (1969) solved numerically a problem concerning the non-linear interactions of the mean flow with the fluctuations in fluid velocity caused by the thermal wave, while the interactions of perturbation quantities were neglected.

It will be shown in §2 that a thermal wave of wave-number k causes a fluid motion consisting of waves of wave-numbers mk, m being any positive integer. When the fluid speed is comparable to the flame speed, the mean flow and the flows of different wave-numbers are of the same order of magnitude, and their mutual interactions cannot be neglected. In fact, Schubert's formulation is valid only when the fluid motion is small compared to the flame speed, while a small effect from the interaction between the mean flow and the flow of wave-number k is included.

The assumption of zero Prandtl number by Stern & Schubert is non-realistic. We will show in §5 that Stern's solution does not converge at low rotational speeds of the thermal source.

Davey's analysis is correct. However, his results were presented only for the limiting cases in which the frequency of flame rotation was either very low or very high. The purpose of the present work is to fill in this gap, within which the fluid speed is much greater and reaches a maximum, and to show the change in mean velocity profile as the frequency increases. Furthermore, the effect of thermal boundary conditions on the flow and shear stress will be examined. It was found in the previous analyses that the induced fluid motion does not exert a net shear force on either plate when the fluid is bounded by two parallel plates. We will show that this is the result of the assumed thermal boundary conditions. An example will be given showing that their conclusion is not true in general.

2. A linearized analysis

Similar to the previous analytical works, the cylindrical annulus of large mean radius is approximated by a two-dimensional channel of depth h supported below by a horizontal plate. The upper surface can either be a flat plate or the free surface of a liquid filling the channel. The x axis is taken to be horizontal at the mid-depth of the fluid and the z axis is taken as the upward normal to the bottom plate. We consider a temperature fluctuation T' of wave form $e^{ik(x+Ut)}$ travelling in the negative x direction with uniform speed U. The mean pressure, density and temperature of the originally stationary fluid are p_0, ρ_0 and T_0, respectively. The temperature perturbation causes fluctuations in pressure, density, horizontal and vertical velocities which are denoted respectively by p', ρ', u', and w'. The mean fluid motion at any height is evaluated by extracting the mean part of u' over one wavelength on that level. The average mean velocity of the fluid, V, is defined as the average value of the mean motion across any section of the channel. The kinematic viscosity and the thermometric conductivity of the fluid are ν and κ, respectively, and are assumed to be constant.

The governing equations can be non-dimensionalized by introducing the following dimensionless variables:

$$\left.\begin{aligned} &\xi = kx, \quad \zeta = z/h, \quad \tau = kUt, \\ &p = p'/\rho_0 gh, \quad T = T'/\Delta T, \quad u = u'/U, \quad w = w'/khU, \end{aligned}\right\} \tag{1}$$

where g is the gravitational acceleration and ΔT is the amplitude of the temperature wave, or the difference between the flame and the mean fluid temperatures. The amplitude of the density fluctuation will be denoted by $\Delta\rho$. By using the Boussinesq approximation and by assuming that the channel depth is much less than the mean radius of the annulus, or $kh \ll 1$, we obtain the simplified continuity and Navier–Stokes equations:

$$\frac{\partial u}{\partial \xi}+\frac{\partial w}{\partial \zeta} = 0, \tag{2}$$

$$\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial \xi}+w\frac{\partial u}{\partial \zeta} = -\Lambda\frac{\Delta\rho}{\rho_0}\frac{\partial p}{\partial \xi}+\frac{1}{R_e}\frac{\partial^2 u}{\partial \zeta^2}, \tag{3}$$

and

$$\frac{\partial p}{\partial \zeta} = T. \tag{4}$$

The dimensionless parameter $\Lambda = gh/U^2$ is the inverse of a Froude number, and $R_e = kh.Uh/\nu$ is a Reynolds number, both based on the speed of the heat source. Equation (4) states that in a shallow channel the perturbed pressure distribution is determined alone by the perturbation in hydrostatic pressure, resulting from the density variation caused by the travelling thermal wave. Equation (3) indicates that this pressure fluctuation, also in the form of a travelling sinusoidal wave, is the driving mechanism of the fluid motion.

With the further assumption that heat is transferred primarily by conduction, and heat convection and dissipation can be neglected, the energy equation is simplified to

$$\frac{\partial^2 T}{\partial \zeta^2}-\Omega\frac{\partial T}{\partial \tau} = 0. \tag{5}$$

$\Omega = \omega h^2/\kappa$ is a non-dimensionalized frequency where $\omega = kU$ is the circular frequency of the rotating heat source. It relates to the Reynolds number through the equation $\Omega = PR_e$, where $P = \nu/\kappa$ is the Prandtl number.

By inspecting (3) and (4), it is clear that a thermal wave of the form

$$T = \mathscr{R}[T_1(\zeta)\, e^{i(\xi+\tau)}] \tag{6}$$

will produce a fluid motion with components

$$u = u_0(\zeta)+\sum_{m=1}^{\infty}\mathscr{R}[u_m(\zeta)\, e^{im(\xi+\tau)}] \tag{7}$$

and

$$w = \sum_{m=1}^{\infty}\mathscr{R}[w_m(\zeta)e^{im(\xi+\tau)}], \tag{8}$$

where $\mathscr{R}$ is used to denote the real part of a complex function. $\mathscr{I}$ will be used later to denote the imaginary part. To obtain a solution for the motion, one has to solve a set of non-linear equations. However, if we assume that the induced fluid motion is slower than the speed of the heat source, u and w are quantities less than unity and the equation (3) can be linearized. By substituting (6), (7) and (8) together with

$$p = \sum_{m=1}^{\infty} \mathscr{R}[p_m(\zeta)\, e^{im(\xi+\tau)}] \tag{9}$$

into (2)–(5) and grouping terms of the same orders, we obtain:

$$iu_1 + \frac{dw_1}{d\zeta} = 0, \tag{10}$$

$$\frac{d^2u_1}{d\zeta^2} - iR_e u_1 = iR_e \Lambda \frac{\Delta\rho}{\rho_0} p_1, \tag{11}$$

$$\frac{d^2u_0}{d\zeta^2} = \tfrac{1}{4}R_e \frac{d}{d\zeta}(u_1\tilde{w}_1 + \tilde{u}_1 w_1), \tag{12}$$

$$\frac{dp_1}{d\zeta} = T_1, \tag{13}$$

and

$$\frac{d^2T_1}{d\zeta^2} - i\Omega T_1 = 0, \tag{14}$$

where a tilde denotes the complex conjugate. Equation (12) shows that u_0 is generated by the Reynolds stress, and is one order smaller than u_1 or w_1. The order of magnitude of u_2 (or w_2) is comparable to that of u_0. Because they are not as interesting as the mean motion to us, they are ignored in the present analysis together with the other smaller quantities.

The general solution of (14) has the form

$$T_1 = a_1 \cosh \lambda\zeta + a_2 \sinh \lambda\zeta, \tag{15}$$

where $\lambda = (i\Omega)^{\frac{1}{2}}$. When the boundary conditions $T_1 = 1$ at $\zeta = \pm\frac{1}{2}$ are chosen, the constants have the values

$$a_1 = \frac{1}{\cosh \frac{1}{2}\lambda}, \quad a_2 = 0. \tag{16}$$

This is the case considered by Davey (1967) in his closed problem. Physically it corresponds to the situation that the upper and lower plates are heated by two aligned identical heat sources, moving in the same direction and at the same speed.

Boundary conditions closer to those in the experiments mentioned in § 1 were used by Davey in his open problem. They are $T_1 = 1$ at $\zeta = -\frac{1}{2}$ and $dT_1/d\zeta = 0$ at $\zeta = \frac{1}{2}$, corresponding to a heated lower plate and an insulated upper surface, which give the result

$$a_1 = \frac{\cosh \frac{1}{2}\lambda}{\cosh \lambda}, \quad a_2 = -\frac{\sinh \frac{1}{2}\lambda}{\cosh \lambda}. \tag{17}$$

The effect of thermal boundary conditions on fluid motion will be studied in the following section.

By use of (15), the solutions of (13), (11) and (10) are

$$p_1 = \frac{1}{\lambda}\left(a_1 \sinh \lambda\zeta + a_2 \cosh \lambda\zeta + \frac{c_1}{P-1}\right), \tag{18}$$

$$u_1 = \frac{i\Lambda}{\lambda(P-1)} \frac{\Delta\rho}{\rho_0} \frac{d\phi}{d\zeta}, \tag{19}$$

and
$$w_1 = \frac{\Lambda}{\lambda(P-1)} \frac{\Delta\rho}{\rho_0} \phi(\zeta), \tag{20}$$

where
$$\alpha = (iR_e)^{\frac{1}{2}}$$

and
$$\phi(\zeta) = \frac{1}{i}\left(\frac{a_1}{\lambda}\cosh\lambda\zeta + \frac{a_2}{\lambda}\sinh\lambda\zeta + \frac{c_2}{\alpha}\cosh\alpha\zeta + \frac{c_3}{\alpha}\sinh\alpha\zeta - c_1\zeta + c_4\right). \tag{21}$$

Substituting (19) and (20) into (12) and using (10), we obtain

$$u_0 = \frac{\Lambda^2}{2P(P-1)^2}\left(\frac{\Delta\rho}{\rho_0}\right)^2 \mathscr{I}\left(\int \phi \frac{d\tilde{\phi}}{d\zeta}\, d\zeta + c_5\zeta + c_6\right). \tag{22}$$

The six constants, c_i, are to be determined by the boundary conditions for the flow field. By the definition of average mean velocity, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u_0\, d\zeta = V/U. \tag{23}$$

3. Fluid bounded by two parallel plates

In this case the boundary conditions are $\phi = d\phi/d\zeta = 0$ at $\zeta = \pm\frac{1}{2}$, which give the values

$$c_1 = a_2 \cosh \tfrac{1}{2}\lambda + c_3 \cosh \tfrac{1}{2}\alpha, \tag{24}$$

$$c_2 = -a_1 \frac{\sinh \frac{1}{2}\lambda}{\sinh \frac{1}{2}\alpha}, \tag{25}$$

$$c_3 = -a_2 \frac{\cosh \frac{1}{2}\lambda - 2\lambda^{-1} \sinh \frac{1}{2}\lambda}{\cosh \frac{1}{2}\alpha - 2\alpha^{-1}\sinh\frac{1}{2}\alpha}, \tag{26}$$

and
$$c_4 = -a_1 \lambda^{-1} \cosh \tfrac{1}{2}\lambda - c_2 \alpha^{-1} \cosh \tfrac{1}{2}\alpha. \tag{27}$$

Two sets of thermal boundary conditions are considered: (i) Assuming both plates are traversed by identical heat sources, the expressions (16) are used for a_1 and a_2. In this case our w_1 in (20) reduces to that obtained by Davey except a sign difference, resulting from a missing negative sign on the right-hand side of Davey's equation (14). However, this error did not appear in his expression for the mean velocity obtained from the product of w_1 and $d\tilde{w}_1/d\zeta$.

Upon substitution of (16) into (24)–(27) and then into (21), ϕ becomes an even function in ζ, and so does the integral in (22). Applying the boundary conditions that $u_0 = 0$ at $\zeta = \pm\frac{1}{2}$, we obtain

$$c_5 = 0 \tag{28}$$

and
$$c_6 = -\left(\int \phi \frac{d\tilde{\phi}}{d\zeta}\, d\zeta\right)_{\zeta=\frac{1}{2}}. \tag{29}$$

The mean flow therefore does not exert shear stresses on the plates, because $\phi\, d\tilde{\phi}/d\zeta$ vanishes there and $c_5 = 0$. This is true in general if the temperature is an even function in ζ, i.e. if it is symmetric about the centre of the channel. The same conclusion was reached by Stern, when considering a temperature profile uniform across the channel.

(ii) If the upper surface is replaced by an insulated plate, the expressions (17) are used for a_1 and a_2. The temperature profile and therefore the function ϕ are no longer symmetric about the centre of the channel, and c_5 becomes finite after applying the boundary conditions for u_0. In this case (22) shows that the shear stresses exerted by the mean flow on the upper and lower plates are opposite to each other and have the same magnitude proportional to c_5. Considering the plates as one unit, the net shear force on the whole system is still zero.

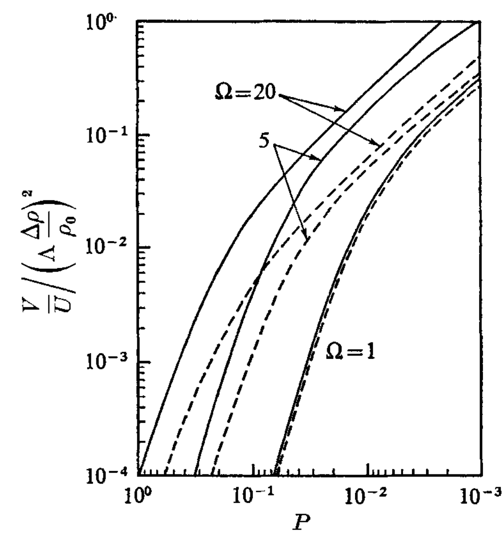

FIGURE 1. The variation of average mean velocity with Prandtl number for some values of Ω. —, identical conditions on both plates; - - -, upper plate insulated.

The general form of the average mean velocity is found to be a strong function of Prandtl number. Their relationship is plotted in figure 1 for fixed values of Ω under different thermal boundary conditions. It shows that the average velocity increases rapidly with decreasing Prandtl number, and the velocity in a channel with an insulated plate is always slower than that with two heated plates, when the other variables are the same.

The average mean velocity is plotted against Ω in figure 2 for different fluid media. We use the approximated values $P = 0{\cdot}04$ for mercury, $0{\cdot}7$ for air, and 7 for water (Landau & Lifshitz 1959). If the thermal boundary conditions are specified, for each fluid there exists an optimum Ω at which the average flow speed reaches a maximum. The average fluid motion is found to be always in a direction opposite to the motion of the heat source.

The mean velocity profiles in mercury at $\Omega = 1$, 10, 100 and 1000 are shown in figure 3. The profiles are symmetric about $\zeta = 0$ when both plates are heated. In the case of an insulated upper plate, the flow near the bottom becomes reversed and moves with the travelling heat source. The profiles are inverted if the upper plate is heated and the lower one insulated, from the fact that u_0 is not affected by the sign of g. At lower frequencies the peaks of the non-symmetric profiles are

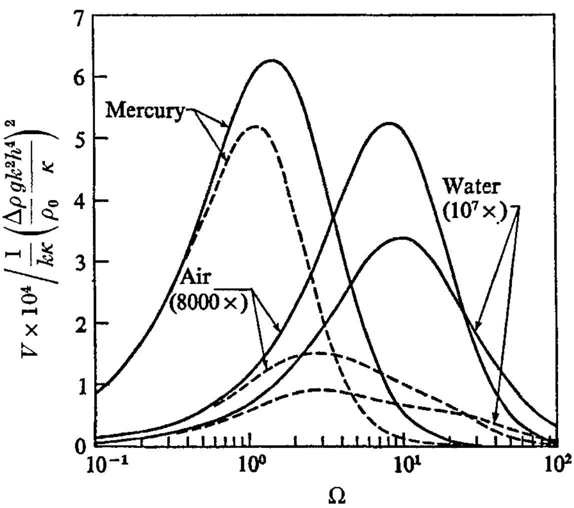

FIGURE 2. The variation of average mean velocity with frequency Ω for mercury, air and water. —, identical conditions on both plates; - - -, upper plate insulated.

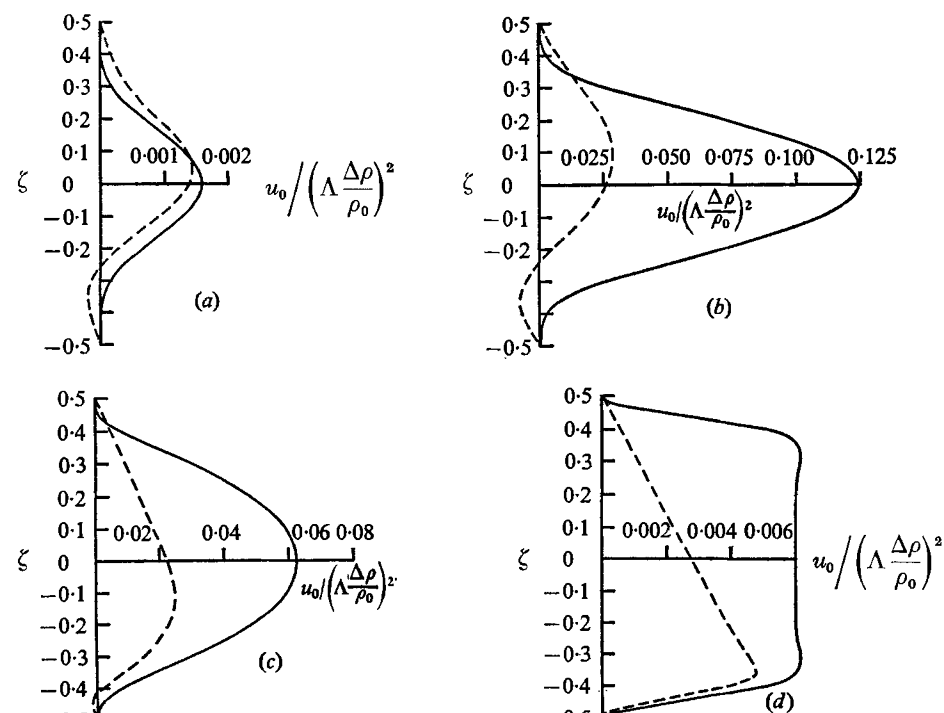

FIGURE 3. The mean velocity profiles in mercury. —, identical conditions on both plates; - - -, upper plate insulated. (a) $\Omega = 1$, (b) $\Omega = 10$, (c) $\Omega = 100$, (d) $\Omega = 1000$.

on the side of the insulated plate, and move toward the other side at higher frequencies. At $\Omega = 1000$, or $R_e = 25{,}000$ for mercury, the flows are of boundary-layer type. At this high frequency, it is interesting to note that, in the symmetric velocity profile, the maximum does not occur at the centre as in the low-frequency cases.

4. Fluid bounded by a plate and a free surface

Fultz *et al.* and Stern used water in their experiments, with the upper surface exposed to the atmosphere. In the experiment of Schubert & Whitehead, mercury was used and was covered with a layer of water. In all cases the fluid above was much lighter than that below, and the working fluid can be regarded as contained between a horizontal plate and an upper free surface.

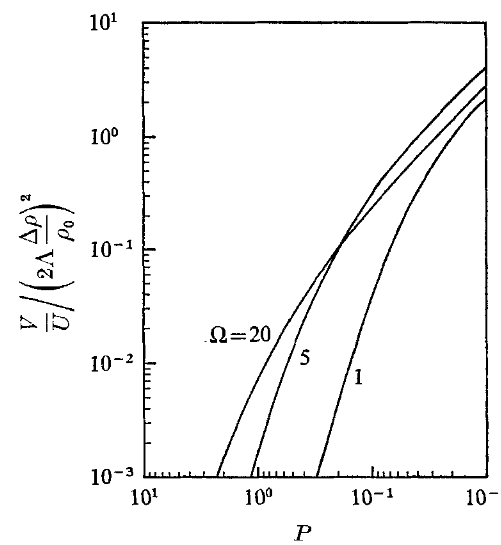

FIGURE 4. With an undeformed upper free surface, the variation of average mean velocity with Prandtl number for some values of Ω.

The thermal boundary conditions were discussed by Davey; they can be approximated by the conditions that the plate is traversed by the heat source and the free surface is thermally insulated. If we assume that the deformation of the free surface is very small and can be neglected, the boundary conditions are $T_1 = 1$ at $\zeta = -\frac{1}{2}$ and $dT_1/d\zeta = 0$ at $\zeta = \frac{1}{2}$.

The velocity boundary conditions at the plate are $\phi = d\phi/d\zeta = 0$ at $\zeta = -\frac{1}{2}$, and those at the free surface are $\phi = d^2\phi/d\zeta^2 = 0$ at $\zeta = \frac{1}{2}$, obtained from the requirement of zero stress and by neglecting the surface deformation there.

This problem is equivalent to one considering only the lower half of the fluid contained between two parallel plates traversed by two identical heat sources. The conditions at the horizontal surface through the centre of the channel are exactly the same as those at an undeformed free surface. The solutions for temperature and velocities for the present problem are deduced immediately, by replacing the channel height with $2h$, from the solutions obtained in the previous section for the case of a symmetric temperature profile. The results are then expressed in a co-ordinate system with the origin shifted to the free surface.

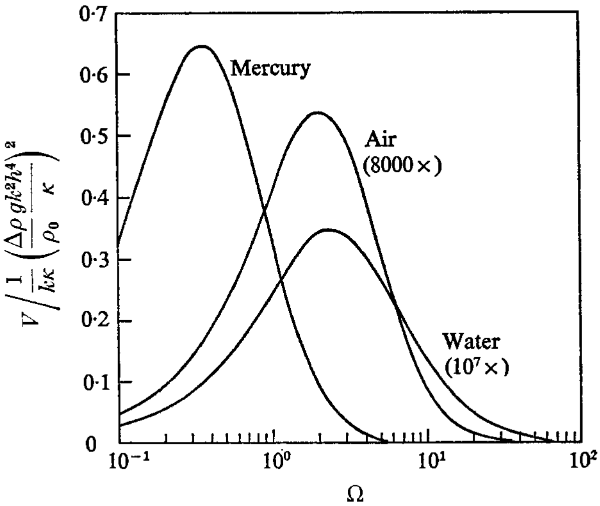

FIGURE 5. With an undeformed upper free surface, the variation of average mean velocity with Ω for mercury, air and water.

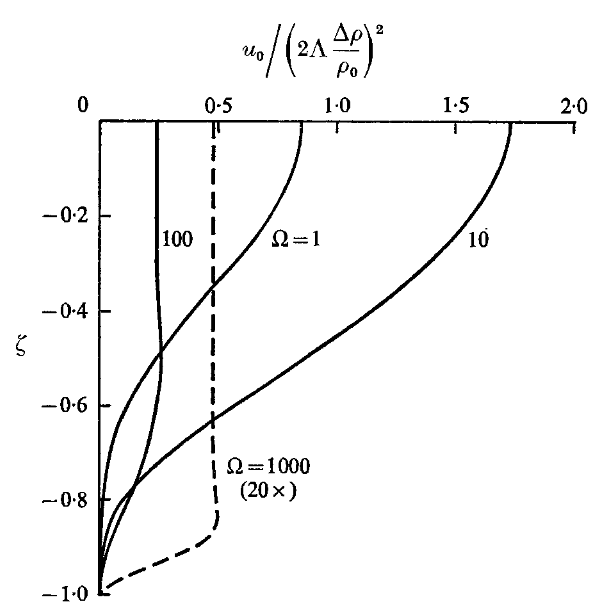

FIGURE 6. With an undeformed upper free surface, the mean velocity profiles in mercury at $\Omega = 1$, 10, 100 and 1000.

The average mean velocity of the fluid is plotted against Prandtl number in figure 4 for $\Omega = 1$, 5 and 20. In comparison with figure 1, it shows that this velocity is one order greater than the value computed for a rigid upper surface. Figure 5 shows the variation of average mean velocity with circular frequency, plotted for mercury, air and water. (The solution for air in a channel with a free surface is non-realistic. It is included here for the purpose of comparison with the corresponding curves in figure 2.) When replacing the upper plate by a free surface, the maximum average mean velocity not only is increased by three orders, but also occurs at a lower frequency.

The mean velocity profiles in mercury at $\Omega = 1$, 10, 100 and 1000 are shown in figure 6. The boundary-layer nature shows again in the flow at high circular frequencies. Shear stress due to the mean flow is zero at both the plate and the free surface.

5. Comparison with previous results

Our analysis shows that the average fluid motion is always in the sense opposite to the motion of the heat source, and its magnitude increases rapidly with decreasing Prandtl number, which agrees qualitatively with the conclusion

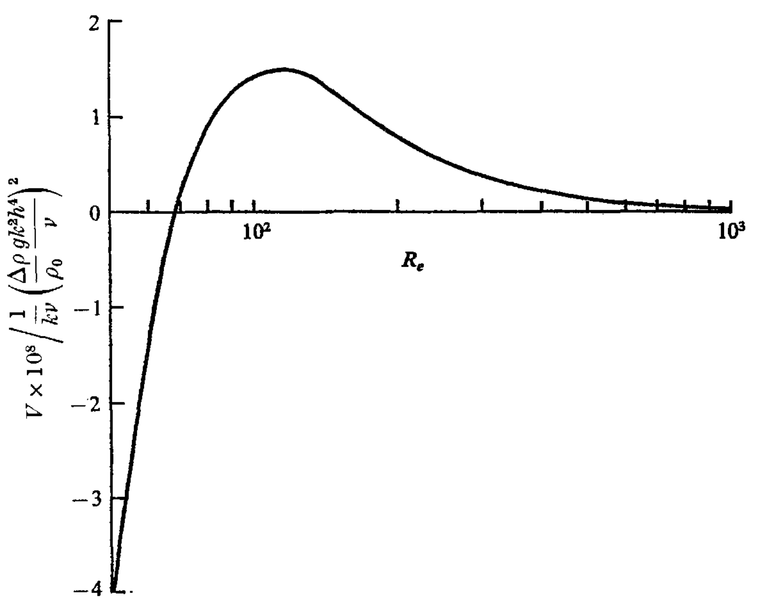

FIGURE 7. The variation of average mean velocity with Reynolds number, based upon Stern's solution for a vanishing Prandtl number.

based upon the observations in water (Fultz *et al.* 1959, Stern 1959) and in mercury (Schubert & Whitehead 1969). Our solid $\Omega = 1$ curve in figure 1 is similar to that obtained by Schubert & Whitehead through numerical integration of the equations.

Stern's solution for a fluid with infinite thermal conductivity indicated, however, that the average flow would move with the flame at low circular frequencies. To examine that solution in detail, the average mean velocity is computed, based upon his expression for the Reynolds stress, and is plotted in our notation as a function of Reynolds number in figure 7. This function becomes negative when R_e is approximately below 68, which was correctly estimated by Stern. From the fact that V does not converge for decreasing R_e, Stern's solution for

$P = 0$ may be regarded as rather non-realistic. Re-examination of the asymptotic expression for V at low R_e reveals that its absolute value is proportional to R_e^{-1} instead of R_e as obtained by Stern.

In the problem concerning a fluid contained between two plates with symmetric temperature profiles, our solution in § 3 is equivalent to that obtained by Davey. But Davey did not show the velocity profiles, and presented only expressions for average mean velocity in the large and small frequency ranges. Fluid speeds in these ranges are very small compared to the maximum value, as can be seen in figure 2, and are relatively unimportant.

In his open problem, Davey used the boundary conditions at the free surface (expressed in our notation)

$$\frac{d^2\phi}{d\zeta^2} = 0 \tag{30}$$

and

$$\frac{d^3\phi}{d\zeta^3} - \alpha^2\frac{d\phi}{d\zeta} + \alpha^2\Lambda\phi = 0. \tag{31}$$

If $\Lambda \gg 1$ and the frequency is not too low so that $|\alpha^2\Lambda| \gg 1$, which are the conditions in all the experiments performed, (31) reduces to $\phi = 0$. In this case, Davey's conditions are the same as those used in our § 4 for an undeformed free surface. The curves shown in figure 5 are in good approximation except their lower frequency parts, which would become negative according to Davey's solution.

In the experiments performed by Fultz *et al.* (1959) and Stern (1959), Ω ranges from 1100 to 9900 (see Davey's table 1). It is much too high compared to 2·5, the optimum value for water at which the average velocity is maximum, as indicated by figure 5. The value of Ω in Schubert & Whitehead's experiment with mercury is about 0·5, which is close to 0·35, the optimum value according to our linearized theory. In their experiment fluid speed was faster than the flame speed, the optimum value of Ω actually should be determined from a non-linear analysis.

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